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Path integral quantisation and coherent states

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Abstract. Generalised coherent states are used in the quantum mechanical study of physical systems with homogeneous phase spaces, for which there is a group theoretical approach to quantisation. New sets of coherent states are introduced based on dynamical subalgebras and also with fiducial vectors in the rigged Hilbert space. We also examine the path integral expression for coherent state transition amplitudes from a group theoretical point of view.

1. Introduction

This paper considers the problem of constructing path integrals in situations where the phase space is not a vector space but a more general homogeneous symplectic manifold S. Our work follows on previous results of Klauder and collaborators who, in a series of interesting papers, have developed the theory of 'continuous representation path integrals' (Klauder and Daubechies 1985 and references therein).

For homogeneous symplectic manifolds S of the form G/H, with G a Lie group and H a closed subgroup, Isham (1984) has emphasised that there is a natural way of quantising by exploiting the representation theory of a group $\mathscr{G}$ (related to G) called hereafter the *canonical group*, assuming that it acts on S through symplectic diffeomorphisms with generators on G/H that are Hamiltonian vector fields (see, for instance, Kostant 1970, Guillemin and Sternberg 1984, Abraham and Marsden 1978).

Briefly, the generators of G on $S = \mathrm{G/H}$ are

$$\left\{\gamma_A \,\middle|\, \gamma_A{}^{(s)}\psi = \frac{\mathrm{d}}{\mathrm{d}t}\psi[\exp(-tA)s]\right|_{t=0}, \mathrm{A} \in L(\mathrm{G}), s \in S\Bigg\} \tag{1}$$

for $\psi \in C^{\infty}(S)$ and $\mathrm{A} \in L(\mathrm{G})$, the Lie algebra of G. Because the generators are Hamiltonian, there are smooth functions $\{P_A\}$ defined up to a constant such that

$$i_{\gamma_A}\omega = \mathrm{d}P_A \tag{2}$$

with ω the symplectic form on S. The functions $\{P_A\}$, for a suitable choice of constants, realise $L(\mathrm{G})$ or a central extension of it as a subalgebra of the Poisson algebra. The resulting Lie algebra can be exponentiated to define the canonical group $\mathscr{G}$ (which is different from G if a central extension term is present in the Poisson algebra of the P_A).

The next step involves the investigation of the unitary irreducible representations of $\mathscr{G}$, as in this group theoretical approach to quantisation they will provide a set of

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appropriate quantum theories associated with the classical phase space. One may consider the groups covering $\mathscr{G}$ since these furnish representations of the Lie algebra as well. This is the analogue of the Heisenberg-Weyl group for quantisation on $S = R^{2n}$. Once a representation has been chosen on physical grounds, say $\hat{U}$ on $\mathscr{H}$, from Stone's theorem we can define a self-adjoint operator $\hat{A}$ on some dense domain of $\mathscr{H}$ corresponding to P_A, namely

$$\hat{A} = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}t} \hat{U}(\exp tA)\bigg|_{t=0}. \tag{3}$$

The set of operators $\{\hat{A}, A \in L(\mathscr{G})\}$ is a sufficient one on which to base quantisation in the sense that any other relevant operators can be written in terms of these (the fundamental quantum observables).

These features can be illustrated by considering the class of theories with configuration space $Q = S^1$ (and hence $S = T^*S^1 = S^1 \times R$). The appropriate canonical group is the two-dimensional Euclidean group $\mathrm{E}(2) = R^2 \circledS \mathrm{SO}(2)$ (Isham 1984) with fundamental observables $S = \sin\theta$, $C = \cos\theta$, $J = p$ (in the natural θ, p coordinates) with Poisson algebra

$$\{C, S\} = 0 \qquad \{J, S\} = -C \qquad \{J, C\} = S \tag{4}$$

defined with respect to the natural symplectic form on T^*S^1. All the representations of E(2) can be 'induced' in the Mackey (1968) sense. They are labelled by a parameter $\lambda \geqslant 0$ and are realised on $\mathscr{H} = L^2(S^1, \mathrm{d}\theta/2\pi)$. Effects of non-trivial $\pi_1(S^1)$ come from representations of the covering groups of E(2) and reproduce the standard θ effects (see, for instance, Schulman 1981, ch 23). For ψ in $\mathscr{H}$ we have

$$\begin{aligned} &(\hat{J}\psi)(\theta) = -\mathrm{i}(\mathrm{d}/\mathrm{d}\theta)\psi(\theta) \qquad (\hat{C}\psi)(\theta) = \lambda \cos\theta\psi(\theta) \\ &(\hat{S}\psi)(\theta) = \lambda \sin\theta\psi(\theta) \end{aligned} \tag{5}$$

in terms of which other observables may be written.

A further example is the phase space S^2 endowed with symplectic form $\omega = \sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\psi$ in the usual spherical coordinates (this has been previously studied in a geometric quantisation context (see Sniatycki 1980)). The natural group to consider is SO(3) which acts transitively on S^2 and leaves ω invariant. The functions

$$L_1 = \sin\theta \cos\psi \qquad L_2 = \sin\theta \sin\psi \qquad L_3 = \cos\theta \tag{6}$$

represent the SU(2) Lie algebra under the Poisson bracket associated with ω. From the representation theory of SU(2) we know that its unitary irreducible representations, $\hat{U}^{(\alpha)}$, are realised on $C^{2\alpha+1}$ where α is positive integer or half-integer. The angular momentum generators $\{\hat{L}_i^{(\alpha)}\}$ correspond to the functions $\{L_i\}$ in (6). This observation was made by Klauder (1979) where it is derived from a semiclassical approximation of a path integral. In this example, the group SU(2) is the canonical group associated with a phase space that is not T^*Q of a configuration space Q. All representations $\hat{U}^{(\alpha)}$ have a physical interpretation.

It is important to emphasise that the path integral formalism for homogeneous symplectic manifolds is not as straightforward as the usual derivation (Schulman 1981, ch 1) might lead one to believe. The reason is that these conventional derivations are intimately connected with the spectral resolutions of the generators of the Weyl-Heisenberg group and imply that the manifold in question admits the usual Fourier analysis.

In the next section we discuss the group theoretic coherent states that we shall use in our path integral investigations. They have been introduced by Klauder (1963) and subsequently studied by Perelomov (1972). They were used in the work of Yaffe (1982) in discussing the classical limits of quantum theories as well as in Lieb (1973), Fuller and Lenard (1979a, b) and Simon (1980) who discuss the classical limits of quantum spin partition functions. Here, we generalise their notion by allowing the fiducial cyclic vector that generates them to belong to a rigged Hilbert space. We also apply the standard definition to dynamical subalgebras.

In § 3 we discuss path integrals that are obtained with the use of these states showing how some of the structures introduced by Klauder (1978, 1979) follow as geometrical objects on a coset space of the canonical group.

2. Group theoretic coherent states

2.1. Construction

The considerations of this section are valid for any irreducible unitary representation of a locally compact group acting on a Hilbert space $\mathscr{H}$, but they are specifically applied to representations of the canonical group ($\mathscr{G}$). We proceed by choosing a fiducial vector $|z_0\rangle \in \mathscr{H}$. We note that the set spanned by $\{\hat{U}(a)|z_0\rangle,\ a \in \mathscr{G}\}$ is dense in $\mathscr{H}$ (if it were not, it would span an invariant subspace of it) so the vector $|z_0\rangle$ is cyclic for the representation. The stability group $\mathscr{G}_0$ of the state $|z_0\rangle$ is defined to be the subgroup of $\mathscr{G}$ such that $a \in \mathscr{G}_0$ implies

$$\hat{U}(a)|z_0\rangle = \exp(\mathrm{i}\beta(a)|z_0\rangle \tag{7}$$

with $\beta(a)$ real. Certainly, $\mathscr{G}_0$ contains the centre of $\mathscr{G}$ as a subgroup.

It is clear that in order to obtain a dense set of vectors it suffices to select one representative from each equivalence class in $\mathscr{G}/\mathscr{G}_0$. The group $\mathscr{G}$ is a principal $\mathscr{G}_0$ bundle over $X = \mathscr{G}/\mathscr{G}_0$. This bundle may not be trivial and in that case possesses no continuous global cross sections. However, we may use measurable cross sections with the property of being smooth in some open set U in X. The measure μ on X will be required to be $\mathscr{G}$-left invariant† and such that $\mu(U) = \mu(X)$. If $g: X \to \mathscr{G}$ denotes our cross section then we define a system of coherent states by

$$\mathfrak{S}_X := \{|z\rangle = \hat{U}[g(z)]|z_0\rangle;\ z \in X\}. \tag{8}$$

The space X appears as a parameter set for a dense set of states in $\mathscr{H}$. The subgroup $\mathscr{G}_0$ is crucially dependent on the choice of cyclic vector $|z_0\rangle$ and may not always be of fixed dimensionality. When $\mathscr{G}/\mathscr{G}_0$ is locally diffeomorphic to the phase space S for which $\mathscr{G}$ is the canonical group, the states in $\mathfrak{S}_X$ are called *phase space coherent states.*

We say the representation $\hat{U}$ is *square integrable* with respect to the cyclic vector $|z_0\rangle$ and the measure μ if

$$M := \int_X \mathrm{d}\mu(z)|\langle z_0|\hat{U}[g(z)]|z_0\rangle|^2 < \infty. \tag{9}$$

The definition is clearly independent of the local section g. For square integrable representations, the resolution of unity

$$\hat{1} = \frac{1}{M}\int_X \mathrm{d}\mu(z)|z\rangle\langle z| \tag{10}$$

† An example of a left $\mathscr{G}$-invariant measure μ is the volume form associated to a left invariant metric on X.

follows from the observation that the right-hand side commutes with all $\hat{U}(a)$, so by Schur's lemma it is a multiple of the unit operator. The normalisation M is given by the integral in (9).

The condition of square integrability has been studied (when $\mathscr{G}_0$ is the centre of $\mathscr{G}$) for unimodular nilpotent groups (Moore and Wolf 1973) and real semisimple groups (Harish-Chandra 1956). For non-unimodular groups, partial results exist as well (Tatsuuma 1972). Square integrability is clearly guaranteed for compact groups.

An interesting variation of the coherent state theme is to let $|z_0\rangle$ belong to the component Φ' of a rigged Hilbert space, $\Phi \subset \mathscr{H} \subset \Phi'$†. This allows the construction of coherent states based on distributions as fiducial vectors. A completeness relation like (10) may then follow as a consequence of a spectral theorem (rather than Schur's lemma), provided they are eigenstates of an operator with continuous spectrum.

Coherent states need not transform equivariantly with the group action but rather they pick a phase factor (Perelomov 1972), i.e.

$$\hat{U}(a)|z\rangle = \exp(\mathrm{i}\gamma_z(a)|az\rangle. \tag{11}$$

The $\gamma_z(a)$ have the 1-cocycle property

$$\gamma_z(ab) = \gamma_{bz}(a) + \gamma_z(b) \tag{12}$$

but it is often not a genuine element of $Z^1(\mathrm{G}, C(X))$ since it is defined with respect to the non-continuous (but measurable) cross section g.

2.2. *Evolution of coherent states*

For a physical system with canonical group $\mathscr{G}$ and $\hat{U}(a)$ an appropriate unitary irreducible representation, we construct the system $\mathfrak{S}_X$ of coherent states based on a cyclic vector $|z_0\rangle$. If the Hamiltonian of the system is $\hat{H}$, the vector $|z\rangle$ in $\mathfrak{S}_X$, then $|z\rangle_t := \exp(-\mathrm{i}t\hat{H}/\hbar)|z\rangle$ will not generally be in $\mathfrak{S}_X$. We say the evolution is *exact* when $|z\rangle_t \in \mathfrak{S}_X$ for all t (modulo phases). This is the case, for instance, if $\hat{H}$ is a linear combination of the generators of $\mathscr{G}$ in the representation that we consider. For exact systems, the quantum evolution may be equivalently described with a trajectory in the parameter space $X = \mathscr{G}/\mathscr{G}_0$. *$\mathscr{G}$-dynamical* or *Schrödinger subalgebras* provide a natural ground for applying the coherent state idea. These subalgebras are defined as follows: let $L(\mathscr{G}_D)$ be a Poisson bracket algebra of functions on S such that (i) $L(\mathscr{G}_D)$ contains $L(\mathscr{G})$ as a subalgebra and the classical Hamiltonian among its generators, and (ii) it can be represented by self-adjoint operators on $\mathscr{H}$ (where $L(\mathscr{G})$ acts irreducibly). These algebras are by definition free from the Van Hove type mathematical obstructions to quantisation‡ (see Chernoff (1981) for a review and Bakas and Kakas (1985) for a discussion in a canonical group context). It is advantageous to introduce coherent states based on $L(\mathscr{G})$-dynamical subalgebras as exactness is assured by construction.

For the Weyl-Heisenberg group W, we can, for example, introduce the harmonic oscillator dynamical subalgebra $L(\mathrm{W} \circledS \mathrm{R})$ (Streater 1967) which has the following

† Φ is a linear dense subset of $\mathscr{H}$ and $\mathscr{H}$ is densely contained in Φ', the dual of Φ. Usually, Φ is constructed to be a nuclear space as then unbounded self-adjoint operators can be diagonalised in Φ' (Gelfand and Vilenkin 1964). For instance $\Phi := \{f \in \mathscr{H} : \mathscr{G} \ni a \to \hat{U}(a)f$ is infinitely differentiable$\}$ is a nuclear space (Nagel 1970) when $\mathscr{G}$ is nilpotent, semisimple with finite centre or of the form $\mathrm{A} \circledS \mathrm{K}$ with A Abelian and K compact.

‡ For the linear case $Q = R$, the maximal finite-dimensional Schrödinger subalgebra is spanned by $\{1, p, q, qp, p^2, q^2\}$.

quantum commutation relations ($\hat{H}_0=\frac{1}{2}(\hat{p}^2+\hat{q}^2)$):

$$[\hat{q},\hat{p}]=\mathrm{i}\hbar \qquad [\hat{q},\hat{H}_0]=\mathrm{i}\hbar\hat{p} \qquad [\hat{p},\hat{H}_0]=-\mathrm{i}\hbar\hat{q}. \tag{13}$$

Introducing coherent states based on this group with $|z_0\rangle$ chosen to be the vacuum state of H_0 one recovers conventional Weyl-Heisenberg coherent states (otherwise known as Glauber states, see Klauder and Sudarshan (1968) for a review).

2.3. *Examples*

In the following examples, coherent states for several groups of physical interest are constructed and their quantum dynamics is investigated ($\hbar=1$ in this section).

2.3.1. Weyl-Heisenberg group. This is the canonical group for systems with $Q=R$. The only infinite-dimensional unitary irreducible representation is realised on $L^2(R,\mathrm{d}x)$ with

$$\hat{U}(q,p,\alpha):=\exp(-\mathrm{i}\alpha\hat{1})\exp[\mathrm{i}(p\hat{q}-q\hat{p})] \tag{14}$$

with $\hat{q}$ represented by multiplication by x and $\hat{p}$ by $-\mathrm{i}(\mathrm{d}/\mathrm{d}x)$.

One can define a set of coherent states by

$$|q,p\rangle=\exp[\mathrm{i}(p\hat{q}-q\hat{p})]|z_0\rangle \tag{15}$$

for any $|z_0\rangle\in L^2(R,\mathrm{d}x)$, where (q,p) are to be interpreted as labels on W$/Z$ where Z is the centre of the group W (generated by 1). However, for a general fiducial vector $|z_0\rangle$ the above set has an exact evolution only for systems whose Hamiltonian is a linear combination of $\hat{p}$, $\hat{q}$ and 1. When $|z_0\rangle$ is allowed to be the zero position eigenstate $|q=0\rangle$ then the q generator factors away as well and we obtain definite position states as rigged coherent states: $|q\rangle=\exp(-\mathrm{i}q\hat{p})|q=0\rangle$. Similarly, the choice $|p=0\rangle$ yields definite momentum states, $|p\rangle=\exp(\mathrm{i}p\hat{q})|p=0\rangle$. The spectral theorem for the $\hat{q}$ and $\hat{p}$ operators provides resolutions of unity in terms of these states. For the harmonic oscillator group W Ⓢ R, we consider the representation

$$\hat{W}(q,p,\alpha,\gamma)=\hat{U}(q,p,\alpha)\exp(-\mathrm{i}\gamma\hat{H}_0). \tag{16}$$

In this parametrisation, the group law is

$$\begin{aligned}(q_1,p_1,\alpha_1,\gamma_1)(q_2,p_2,\alpha_2,\gamma_2)\\ &=(q_1+q_2\cos\gamma_1+p_2\sin\gamma_1,\ p_1+p_2\cos\gamma_1-q_2\sin\gamma_1,\ \alpha_1+\alpha_2\\ &\quad+\tfrac{1}{2}[(q_1p_2-p_1q_2)\cos\gamma_1-(q_1q_2+p_1p_2)\sin\gamma_1],\ \gamma_1+\gamma_2).\end{aligned} \tag{17}$$

Coherent states can be defined by letting $\hat{W}$ act on a fiducial vector and factoring out the phase subgroup. If $\hat{H}_0$ in (16) were the free particle Hamiltonian (also a dynamical algebra of the form L(W Ⓢ R)) then the group law would be

$$\begin{aligned}(q_1,p_1,\alpha_1,\gamma_1)(q_2,p_2,\alpha_2,\gamma_2)\\ &=(q_1+q_2+p_2\gamma_1,\ p_1+p_1,\ \alpha_1+\alpha_2+\tfrac{1}{2}[q_1p_2-p_1q_2-p_1p_2\gamma_1],\ \gamma_1+\gamma_2).\end{aligned} \tag{18}$$

From the group law (17) (resp (18)), the quantum evolution of the harmonic oscillator (resp free particle) can be equivalently described by a curve in the parameter space of coherent states defined by the representation W:

$$q(t)=q\cos t+p\sin t \qquad p(t)=p\cos t-q\sin t \tag{17'}$$

$$q(t)=q+pt \qquad p(t)=p \tag{18'}$$

respectively, which are identified as the classical time evolutions.

Suppose we choose $|z_0\rangle$ to be an eigenstate of the harmonic oscillator Hamiltonian; when (16) acts on $|z_0\rangle$ the γ term will factor away yielding coherent states that are the same as those in (15). When $|z_0\rangle$ is the harmonic oscillator vacuum state, we get Glauber states.

2.3.2. SU(2) states. As we have pointed out, SU(2) is the canonical group for non-relativistic spin: we review the construction of Perelomov (1972) of coherent states for this group.

We choose to use the Euler angle parametrisation of SU(2), namely

$$\hat{U}^{(\alpha)}(\phi, \theta, \rho) = \exp(\mathrm{i}\phi\hat{L}_3^{(\alpha)}) \exp(\mathrm{i}\theta\hat{L}_2^{(\alpha)}) \exp(\mathrm{i}\rho\hat{L}_3^{(\alpha)}) \tag{19}$$

for spin α. Choosing $|z_0\rangle = |\alpha, -\alpha\rangle$ the stability group consists of elements of the form $\exp(\mathrm{i}\rho\hat{L}_3)$, and hence the set of coherent states is

$$|\phi, \theta\rangle_{(\alpha)} := \exp(\mathrm{i}\phi\hat{L}_3^{(\alpha)}) \exp(\mathrm{i}\theta\hat{L}_2^{(\alpha)})|\alpha, -\alpha\rangle \tag{20}$$

parametrised by points on S^2 and the resolution of the identity is

$$\hat{1}_{(\alpha)} = \frac{2\alpha+1}{4\pi} \int_{S^2} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi (|\phi, \theta\rangle\langle\phi, \theta|)_{(\alpha)}. \tag{21}$$

2.3.3. E(2) group. The unitary irreducible representations of E(2) act on $L^2(S^1, \mathrm{d}\theta)$, but they are not square integrable so it is not possible to use the Perelomov technique. Rigged coherent states are still possible by choosing $|z_0\rangle$ to be a $\hat{C}$ (and hence $\hat{S}$) eigenstate. Then 'definite angle' states $|\theta'\rangle = \exp(-\mathrm{i}\theta'\hat{J})|z_0\rangle$ result after factoring out the subgroup generated by C and S. They are rigged† coherent states (eigenstates of the $\hat{C}$ and $\hat{S}$ operators). The spectral theorem for C (or S) yields a resolution of unity

$$\hat{1} = \int \frac{\mathrm{d}\theta}{2\pi} |\theta\rangle\langle\theta|. \tag{22}$$

2.3.4. Affine group, $R \circledS R_+$. This is the canonical group for $Q = R_+$ (Isham 1984) with commutation relations

$$[\hat{q}, \hat{\pi}] = \mathrm{i}\hat{q}. \tag{23}$$

Because R_+ has three orbits in R there are three different unitary irreducible representations (see for instance Mackey 1968). The interesting physical ones are realised on $L^2(R_+, \mathrm{d}x/x)$ and $L^2(R_-, \mathrm{d}x/x)$. The one based on wavefunctions with support on the positive real line has generators

$$(\hat{q}\psi)(x) = x\psi(x) \qquad (\hat{\pi}\psi)(x) = -\mathrm{i}x(\mathrm{d}/\mathrm{d}x)\psi(x). \tag{24}$$

Aslaksen and Klauder (1969) have introduce affine coherent states based on the fiducial vector $z_0(x) = 2x\exp(-x^2)$,

$$|\nu, \lambda\rangle = \exp(-\mathrm{i}\nu\hat{q}) \exp[-\mathrm{i}(\ln\lambda)\hat{\pi}]|z_0\rangle. \tag{25}$$

We next consider a $R \circledS R_+$ dynamical subalgebra that contains q and π and the Hamiltonian given by

$$\hat{H}_\Lambda = \tfrac{1}{2}\hat{\pi}(1/\hat{q})\hat{\pi} + \tfrac{1}{2}\Lambda\hat{q} \qquad \text{with } \Lambda > 0 \tag{26}$$

† The Gelfand triplet being $\Phi \subset L^2(S^1) \subset \Phi'$, Φ = differentiable vectors of the representation.

with commutation relations

$$[\hat{q}, \hat{\pi}] = \mathrm{i}\hat{q} \qquad [\hat{q}, \hat{H}_\Lambda] = \mathrm{i}\hat{\pi} \qquad [\pi, \hat{H}_\Lambda] = \mathrm{i}[\hat{H}_\Lambda - \Lambda\hat{q}]. \tag{27}$$

If we parametrise the unitary representation associated with (27) by

$$S(\nu, \lambda, c) = \exp(-\mathrm{i}\nu\hat{q}) \exp[-\mathrm{i}(\ln\lambda)\hat{\pi}] \exp(-\mathrm{i}c\hat{H}_0) \tag{28}$$

with $H_0 = H_{(\Lambda=0)}$ then the group law takes the form

$$(\nu_1, \lambda_1, c_1)(\nu_2, \lambda_2, c_2)$$

$$= \left(\frac{2\nu_1 + 2\lambda_1^{-1}\nu_2 - \nu_2\nu_2 c_1}{2 - \nu_2 c_1}, \left(1 - \frac{\nu_2 c_1}{2}\right)^2 \lambda_1\lambda_2, \frac{2c_2 + 2\lambda_2^{-1}c_1 - \nu_2 c_1 c_2}{2 - \nu_2 c_1}\right). \tag{29}$$

Coherent states based on this dynamical subalgebra have an exact evolution for a system whose Hamiltonian is $\hat{H}_\Lambda$. The quantum trajectory in (ν, λ, c) space is given by

$$\lambda(t) = \lambda \cos^2\left(\frac{t\sqrt{\Lambda}}{2}\right)\left[1 - \frac{\nu}{\sqrt{\Lambda}} \tan\left(\frac{t\sqrt{\Lambda}}{2}\right)\right]^2 \tag{30}$$

$$\nu(t) = \sqrt{\Lambda} \tan\left(\frac{t\sqrt{\Lambda}}{2}\right) + \nu\left\{\cos^2\left(\frac{t\sqrt{\Lambda}}{2}\right)\left[1 - \frac{\nu}{\sqrt{\Lambda}} \tan\left(\frac{t\sqrt{\Lambda}}{2}\right)\right]\right\}^{-1} \tag{31}$$

$$c(t) = c + \frac{2}{\lambda\sqrt{\Lambda}} \tan\left(\frac{t\sqrt{\Lambda}}{2}\right)\left[1 - \frac{\nu}{\sqrt{\Lambda}} \tan\left(\frac{t\sqrt{\Lambda}}{2}\right)\right]^{-1} \tag{32}$$

where $\lambda = \lambda(0)$, $\nu = \nu(0)$, $c = c(0)$. We note that the quantum evolution in this case can be equivalently described by a curve in the (ν, λ, c) parameter space. However, the dynamical group considered here is not a semidirect product extension of the canonical group R Ⓢ R_+ (cf (17) and (18)) and hence the $c(t)$ component of the above time evolution is not trivial. This feature is a peculiarity of the affine group which becomes transparent when coherent states based on dynamical algebras are considered.

3. Path integral quantisation

If meaningful resolutions of unity exist for the system of coherent states $\mathfrak{G}_X$ and moreover the states are overcomplete in the sense that any two of them are never orthogonal (except in a set of μ-measure zero), then one can construct continuous representation path integrals following Klauder (1978, 1979). The transition amplitude $\langle z'', t''|z', t'\rangle$ to go from state $|z'\rangle$ at t' to $|z''\rangle$ at t'' is

$$\left\langle z''\left|\left[\exp\left(-\frac{\mathrm{i}\varepsilon\hat{H}}{\hbar}\right)\right]^{N+1}\right|z'\right\rangle = \int \prod_{i=1}^{N} \frac{\mathrm{d}\mu(z_i)}{M} \prod_{j=0}^{N} \left\langle z_{j+1}\left|\exp\left(-\frac{\mathrm{i}\varepsilon\hat{H}}{\hbar}\right)\right|z_j\right\rangle. \tag{33}$$

A skeletonisation of the time interval (t', t'') has been performed where $\varepsilon = (t'' - t')/(N+1)$. In the small ε limit, or equivalent large N limit, we write

$$\left\langle z_{j+1}\left|\exp\left(-\frac{\mathrm{i}\varepsilon\hat{H}}{\hbar}\right)\right|z_j\right\rangle \simeq \langle z_{j+1}|z_j\rangle \exp\left(\frac{-\mathrm{i}\varepsilon H(z_{j+1}|z_j)}{\hbar}\right) \tag{34}$$

with

$$H(z_{j+1}|z_j) = \langle z_{j+1}|\hat{H}|z_j\rangle / \langle z_{j+1}|z_j\rangle. \tag{35}$$

Thus (33) can be written

$$\lim_{N\to\infty} \int \prod_{i=1}^{N} \frac{d\mu(z_i)}{M} \exp \sum_{j=0}^{N} \left[\ln\langle z_{j+1}|z_j\rangle - \left(\frac{i\varepsilon}{\hbar}\right) H(z_{j+1}|z_j) \right]. \tag{36}$$

We shall obtain an expression for the transition amplitude (36) in a way that reflects the natural $\mathscr{G}$ action on $X = \mathscr{G}/\mathscr{G}_0$. By the transitivity of the $\mathscr{G}$ action there is a group element a_j (modulo the isotropy group of z_j) that translates z_j into z_{j+1}, i.e.

$$z_{j+1} = l_{a_j}(a_j) = a_j z_j. \tag{37}$$

Because the canonical group is constructed as the exponential of the Poisson Lie algebra of the P_A, we can associate to a_j a Lie algebra element A_j such that

$$a_j = \exp A_j \tag{38}$$

and to which we can associate a fundamental vector field $\sigma(A_j)_z$ by

$$\sigma(A_j)_z \psi = \frac{d}{dt} \psi[(\exp tA_j z)]|_{t=0} \tag{39}$$

with the property

$$\exp \sigma(A_j)_z \psi = \psi(a_j z) \tag{40}$$

for any function ψ on X. In particular

$$(\exp - \sigma(A_j))_{z_{j+1}} \psi = \psi(a_j^{-1} z_{j+1}) = \psi(z_j). \tag{41}$$

The vector field $\sigma(A_j)$ can be thought of as the 'difference' between two points z_{j+1} and z_j in a manifold with a general transitive group action rather than an affine structure.

We define $|\zeta\rangle : X \to \mathscr{H}$ by

$$|\zeta(z)\rangle := |z\rangle \tag{42}$$

and introduce, as is usual in path integral discussions, the midpoint $\bar{z}_j$ between z_j and z_{j+1} defined through the properties

$$(\exp \tfrac{1}{2}\sigma(A_j))_{\bar{z}_j} \psi = \psi(z_{j+1}) \qquad (\exp - \tfrac{1}{2}\sigma(A_j))_{\bar{z}_j} \psi = \psi(z_j). \tag{43}$$

In the appendix we show how $\ln\langle z_j|z_{j+1}\rangle$ can be expanded to second order in $\sigma(A_j)$ to give

$$\begin{aligned} &-\langle \bar{z}_j|(\sigma(A_j)_{\bar{z}_j}|\zeta\rangle) \\ &-\tfrac{1}{2}[(\sigma(A_j)_{\bar{z}_j}\langle\zeta|)(\sigma(A_j)_{\bar{z}_j}|\zeta\rangle) \\ &+\langle \bar{z}_j|(\sigma(A_j)_{\bar{z}_j}|\zeta\rangle)\langle \bar{z}_j|(\sigma(A_j)|\zeta\rangle)] + \dots \end{aligned} \tag{44}$$

When (44) above is substituted in (36), a suitable time-sliced version of the path integral is obtained, provided it is sufficient to keep the expansion only to second order in $\sigma(A_j)$.

It is convenient to introduce a 1-form and a metric:

$$\theta_z = -i\hbar\langle z|d|z\rangle \tag{45a}$$

$$-(1/\hbar^2)\rho_z = [\langle z|d|z\rangle \otimes_s \langle z|d|z\rangle + (d\langle z|) \otimes_s (d|z\rangle)] \tag{45b}$$

using d, the exterior differential operator on $U \subset X$ and $\otimes_s$, the symmetric tensor product. With these structures, (44) may be written

$$\frac{i}{\hbar} \theta_{\bar{z}_j}(\sigma(A_j)) - \frac{1}{2\hbar^2} \rho_{\bar{z}_j}(\sigma(A_j), \sigma(A_j)). \tag{46}$$

The 1-form θ_z is not $\mathscr{G}$-invariant. However, using (11) we see that

$$(1_a^*\theta)_z = \theta_z - \mathrm{d}\gamma_z(a) \tag{47}$$

so that $\mathrm{d}\theta$ is in fact $\mathscr{G}$-invariant, thus suggesting a possible identification of θ as a presymplectic form ($\omega = \mathrm{d}\theta$) for X. Again we make the proviso that because the cross section g which defines the states $|z\rangle$ is not necessarily continuous except in a domain $U \subset X$, then θ is not always globally defined. Further, X is not guaranteed to be of even dimensionality (thus not admitting a symplectic form). Using (11), the metric ρ may be verified to be $\mathscr{G}$-invariant within the open set U and it is therefore possible to extend ρ to a global object on all of X. Further, one can show ρ to be positive definite. In particular, if we choose $g(z) = \exp(z^a t_a)$ with $\{t_a\}$ a basis of $L(\mathscr{G})/L(\mathscr{G}_0)$, then ρ at the origin of X (the coset $\mathscr{G}_0$) is

$$\rho_{\mathscr{G}_0} = \hbar^2(\tfrac{1}{2}\langle z_0|(\hat{t}_a\hat{t}_b + \hat{t}_b\hat{t}_a)|z_0\rangle - \langle z_0|\hat{t}_a|z_0\rangle\langle z_0|\hat{t}_b|z_0\rangle)\, \mathrm{d}z^a \otimes_s \mathrm{d}z^b \tag{48}$$

which is clearly positive definite.

We can now write a formal expression for the path integral (36). Denote by $\dot{Z}_j = \sigma(A_j)/\varepsilon$ the average velocity vector between steps; then the continuum version of (40) may be expressed as

$$\int \prod_{t'<t<t''} \frac{\mathrm{d}\mu(z_t)}{M} \exp\left[\frac{\mathrm{i}}{\hbar}\int_{t'}^{t''} \mathrm{d}t\left(\theta_{z_t}(\dot{Z}_t) - \frac{\mathrm{i}\varepsilon}{2\hbar}\rho_z(\dot{Z}_t, \dot{Z}_t) - H(z_t)\right)\right] \tag{49}$$

where we have left a 'small' ε factor in front of the metric.

A rigorous approach to (49) can be formulated by interpreting the metric factor in it together with the $\mathscr{G}$-invariant measures $\Pi\, \mathrm{d}\mu(z_t)$ as defining a Wiener measure for Brownian motion on X with diffusion constant $\hbar/\varepsilon$. The actual amplitude could be recovered by taking the limit $\epsilon \to 0$ (Klauder and Daubechies 1984, 1985).

The path integral for Glauber coherent states assumes the form ($\hbar = 1$),

$$\langle p''q''t''|p'q't'\rangle = \int \prod_{t'<t<t''} \frac{\mathrm{d}p_t\, \mathrm{d}q_t}{2\pi} \exp\left[\mathrm{i}\int_{t'}^{t''} \mathrm{d}t\left((p\dot{q} - \dot{p}q) + \frac{\mathrm{i}\varepsilon}{4}(p^2 + q^2) - H(p, q)\right)\right] \tag{50}$$

whereas the path integral for spin (Klauder 1979, Kuratsuji and Suzuki 1980), now shown to follow from group theoretic quantisation on S^2, is

$$\langle\theta'', \phi'', t''|\theta', \phi', t'\rangle = \int \prod_{t'<t<t''} \frac{\sin\theta_t\, \mathrm{d}\theta_t\, \mathrm{d}\phi_t}{(2\alpha+1)^{-1}(4\pi)} \times \exp\left(\mathrm{i}\int_{t'}^{t''} \mathrm{d}t[\alpha\cos\theta\dot{\phi} + \tfrac{1}{4}\mathrm{i}\varepsilon\alpha(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) - H(\theta_t, \phi_t)]\right). \tag{51}$$

The form $\alpha\cos\theta\, \mathrm{d}\phi$ appearing in the exponent (51) is not globally defined, a not unsurprising fact since S^2 does not admit a presymplectic form because it is compact (it is also not the cotangent bundle of some configuration space). This lack of global definition can be traced back to the definition (20); as is well known, the fibre bundle $\mathrm{U}(1) \to \mathrm{SU}(2) \to S^2 = \mathrm{SU}(2)/\mathrm{U}(1)$ is not trivial so the comments after (47) are applicable.

4. Discussion

We have shown how when the square integrability requirement is satisfied, coherent state path integrals can be constructed for phase spaces that are not linear but carry instead a transitive group action (homogeneous space). These path integrals incorporate the effects of the group theoretic approach to quantisation of the kinematical variables.

The matrix element $H(z_t)=\langle z, t|\hat{H}|z, t\rangle$ (cf (35)) can be evaluated by virtue of the formula

$$\mathrm{e}^{Y}X\,\mathrm{e}^{-Y}=X+[Y, X]+\tfrac{1}{2}[Y,[Y, X]]+\ldots. \tag{52}$$

For exact systems, $H(z_t)=\langle z_0|\hat{H}_t|z_0\rangle$, where $\hat{H}_t$ is a linear combination of the generators of the group that is associated with the coherent states $\mathfrak{G}_X$. In general, Klauder (1967) conjectures that $\lim_{\hbar\to 0} H(z_t)=H_{\mathrm{clas}}$, the classical Hamiltonian of the system in question.

Lastly, we comment on the validity of expansion (44). If the rigorous approach to (49) is valid, i.e. if it is indeed a Wiener path integral, then no terms of higher than quadratic order in $\sigma(A_j)$ will contribute because these are invisible to the (Wiener) measure.

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Appendix

Our objective is to find an expression for $\ln\langle z_{j+1}|z_j\rangle$. Using (43), we see that it can be written as

$$\ln\{[(\exp\tfrac{1}{2}\sigma(A_j)_{\bar{z}_j}\langle\zeta|)][(\exp-\tfrac{1}{2}\sigma(A_j)_{\bar{z}_j}|\zeta\rangle)]\}. \tag{A1}$$

The exponentials above can be expanded to second order in $\sigma(A_j)$ to obtain

$$\ln\{[1+\tfrac{1}{2}(\sigma(A_j)_{\bar{z}_j}\langle\zeta|)|\bar{z}_j\rangle+\tfrac{1}{4}(\sigma(A_j)_{\bar{z}_j}\sigma(A_j)\langle\zeta|)|\bar{z}_j\rangle]$$
$$\times[-\tfrac{1}{2}\langle\bar{z}_j|(\sigma(A_j)_{\bar{z}_j}|\zeta\rangle)+\tfrac{1}{4}\langle\bar{z}_j|(\sigma(A_j)_{z_j}\sigma(A_j)|\zeta\rangle)+\ldots]\}. \tag{A2}$$

Using the fact that $\langle\zeta|\zeta\rangle=1$, it follows that for any vector field Y,

$$(\boldsymbol{Y}_z\langle\zeta|)|z\rangle+\langle z|(\boldsymbol{Y}_z|\zeta\rangle)=0 \tag{A3a}$$

$$2(\boldsymbol{Y}_z\langle\zeta|)(\boldsymbol{Y}_z|\zeta\rangle)+(\boldsymbol{Y}_z\boldsymbol{Y}\langle\zeta|)|z\rangle+\langle z|(\boldsymbol{Y}_z\boldsymbol{Y}|\zeta\rangle)=0 \tag{A3b}$$

we can then rewrite (A3) and expand the logarithm to obtain precisely (44).

Equation (46) follows from the differential geometry identity

$$\boldsymbol{X}_z\phi=\mathrm{d}\phi(\boldsymbol{X}). \tag{A4}$$

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